The connection between field-theory and the equations for material sistems

L. I. Petrova

Moscow State University, Russia, e-mail: ptr@cs.msu.su

Abstract

The existing field theories are based on the properties of closed exterior forms, which correspond to conservation laws for physical fields.

In the present paper it is shown that closed exterior forms corresponding to field theories are obtained from the equations modelling conservation (balance) laws for material sistems (material media).

The process of obtaining closed exterior forms demonstrates the connection between field-theory equations and the equations for material sistems and points to the fact that the foundations of field theories must be conditioned by the properties of equations conservation laws for material sistems.

1. Peculiarities of differential equations for material sistems

Equations for material sistems are equations that describe the conservation laws for energy, linear momentum, angular momentum and mass. Such conservation laws can be named as balance ones since they establish the balance between the variation of a physical quantity and corresponding external action.

[The material system - material (continuous) medium - is a variety (infinite) of elements that have internal structure and interact among themselves. Thermodynamical, gasodynamical and cosmologic system, systems of elementary particles and others are examples of material system. (Physical vacuum can be considered as an analog of such material system.) Electrons, protons, neutrons, atoms, fluid particles and so on are examples of elements of material system.]

The equations of balance conservation laws are differential (or integral) equations that describe a variation of functions corresponding to physical quantities [1-3]. (The Navier-Stokes equations are an example [3].)

It appears that, even without a knowledge of the concrete form of these equations, one can see specific features of these equations and their solutions using skew-symmetric differential forms [4-6].

To do so it is necessary to study the conjugacy (consistency) of these equations.

The functions for equations of material sistems sought are usually functions which relate to such physical quantities like a particle velocity (of elements), temperature or energy, pressure and density. Since these functions relate to one material system, it has to exist a connection between them. This connection is described by the state-function. Below it will be shown that the analysis of integrability and consistency of equations of balance conservation laws for material media reduces to a study the nonidentical relation for the state-function.

Let us analyze the equations that describe the balance conservation laws for energy and linear momentum.

We introduce two frames of reference: the first is an inertial one (this frame of reference is not connected with the material system), and the second is an accompanying one (this system is connected with the manifold built by the trajectories of the material system elements).

The energy equation in the inertial frame of reference can be reduced to the form:

$$\frac{D\psi}{Dt} = A_1$$

where D/Dt is the total derivative with respect to time, ψ is the functional of the state that specifies the material system, A_1 is the quantity that depends on specific features of the system and on external energy actions onto the system. {The action functional, entropy, wave function can be regarded as examples of the functional ψ. Thus, the equation for energy presented in terms of the action functional S has a similar form: $DS/Dt = L$, where $\psi = S$, $A_1 = L$ is the Lagrange function. In mechanics of continuous media the equation for energy of an ideal gas can be presented in the form [3]: $Ds/Dt = 0$, where s is entropy.}

In the accompanying frame of reference the total derivative with respect to time is transformed into the derivative along the trajectory. Equation of energy is now written in the form

$$\frac{\partial\psi}{\partial\xi^1} = A_1 \tag{1}$$

Here ξ^1 is the coordinate along the trajectory.

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial\psi}{\partial\xi^\nu} = A_\nu, \quad \nu = 2, ... \tag{2}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (1) and (2) can be convoluted into the relation

$$d\psi = A_\mu\, d\xi^\mu, \quad (\mu = 1, \nu) \tag{3}$$

where $d\psi$ is the differential expression $d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu$.

Relation (3) can be written as

$$d\psi = \omega \tag{4}$$

here $\omega = A_\mu\, d\xi^\mu$ is the skew-symmetrical differential form of the first degree (the summation over repeated indices is implied).

Relation (4) has been obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form will be a form of the second degree. And in combination with the equation

of the balance conservation law for mass this form will be a form of degree 3. In general case the evolutionary relation can be written as

$$d\psi = \omega^p \tag{5}$$

where the form degree p takes the values $p = 0, 1, 2, 3$. (The relation for $p = 0$ is an analog to that in the differential forms of zero degree, and it was obtained from the interaction of energy and time.)

Since the balance conservation laws are evolutionary ones, the relations obtained are also evolutionary relations, and the skew-symmetric forms ω and ω^p are evolutionary ones.

Relations obtained from the equation of the balance conservation laws turn out to be nonidentical. In the left-hand side of evolutionary relation (4) there is a differential that is a closed form. This form is an invariant object. The right-hand side of relation (4) involves the differential form ω, that is not an invariant object because in real processes, as it is shown below, this form proves to be unclosed.

For the form to be closed the differential of the form or its commutator must be equal to zero. Let us consider the commutator of the form $\omega = A_\mu d\xi^\mu$. The components of the commutator of such a form can be written as follows:

$$K_{\alpha\beta} = \left(\frac{\partial A_\beta}{\partial \xi^\alpha} - \frac{\partial A_\alpha}{\partial \xi^\beta}\right) \tag{6}$$

(here the term connected with the manifold metric form has not yet been taken into account).

The coefficients A_μ of the form ω have been obtained either from the equation of the balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form ω constructed from the derivatives of such coefficients is nonzero. This means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and is not an invariant quantity.

This means that the relation (4) involves not an invariant term. Such a relation cannot be an identical one. Hence, without the knowledge of particular expression for the form ω, one can argue that for actual processes the relation obtained from the equations corresponding to the balance conservation laws proves to be nonidentical. In similar manner it can be shown that general relation (5) is also nonidentical.

{The peculiarities of the evolutionary relation are connected with the differential form that enters into this relation. This is a skew-symmetric form with the basis, in contrast to the basis of exterior form, is a deforming (nondifferentiable) manifold. (About the properties of such skew-symmetric form one can read, for example, in paper [6]). The peculiarity of skew-symmetric forms defined on such manifold is the fact that their differential depends on the basis. The commutator of such form includes the term that is connected with a differentiating the basis. This can be demonstrated by an example of the first-degree skew-symmetric form.

Let us consider the first-degree form $\omega = a_\alpha dx^\alpha$. The differential of this form can be written as $d\omega = K_{\alpha\beta} dx^\alpha dx^\beta$, where $K_{\alpha\beta} = a_{\beta;\alpha} - a_{\alpha;\beta}$ are the components of the commutator

of the form ω, and $a_{\beta;\alpha}$, $a_{\alpha;\beta}$ are the covariant derivatives. If we express the covariant derivatives in terms of the connectedness (if it is possible), then they can be written as $a_{\beta;\alpha} = \partial a_\beta/\partial x^\alpha + \Gamma^\sigma_{\beta\alpha} a_\sigma$, where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. If we substitute the expressions for covariant derivatives into the formula for the commutator components, we obtain the following expression for the commutator components of the form ω:

$$K_{\alpha\beta} = \left(\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta}\right) + (\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta}) a_\sigma$$

Here the expressions $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ entered into the second term are just components of commutator of the first-degree metric form that specifies the manifold deformation and hence equals nonzero. (In commutator of the exterior form, which is defined on differentiable manifold the second term is not present.) [It is well-known that the metric form commutators of the first-, second- and third degrees specifies, respectively, torsion, rotation and curvature.]

The skew-symmetric form in the evolutionary relation is defined in the manifold made up by trajectories of the material system elements. Such a manifold is a deforming manifold. The commutator of skew-symmetric form defined on such manifold includes the metric form commutator being nonzero. (The commutator of unclosed metric form, which is nonzero, enters into commutator (6) of the evolutionary form $\omega = A_\mu d\xi^\mu$.) Such commutator of differential form cannot vanish. And this means that evolutionary skew-symmetric form that enters into evolutionary relation cannot be closed. The evolutionary relation cannot be an identical one. (Nonclosure of evolutionary form and an availability of additional term in this form commutator governs the properties and peculiarities of nonidentical evolutionary relation and its physical importance.)}

Nonidentity of the evolutionary relation means that initial equations of balance conservation laws are not conjugated, and hence they are not integrable. The solutions of these equations can be functional or generalized ones. In this case generalized solutions are obtained only under degenerated transformations.

The evolutionary relation obtained from equations of balance conservation laws for material systems (continuous media) carries not only mathematical but also large physical loading [6,7]. This is due to the fact that the evolutionary relation possesses the duality. On the one hand, this relation corresponds to material system, and on other, as it will be shown below, describes the mechanism of generating physical structures. This discloses the properties and peculiarities of the field-theory equations and their connection with the equations of balance conservation laws for material systems.

Physical significance of nonidentical evolutionary relation.

The evolutionary relation describes the evolutionary process in material system since this relation includes the state differential $d\psi$, which specifies the material system state. However, since this relation turns out to be not identical, from this relation one cannot get the differential $d\psi$. The absence of differential means that the system state is nonequilibrium.

The evolutionary relation possesses one more peculiarity, namely, this relation is a selfvarying relation. (The evolutionary form entering into this relation is defined on the deforming manifold made up by trajectories of the material system elements. This means that the evolutionary form basis varies. In turn, this leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated.)

Selfvariation of the nonidentical evolutionary relation points to the fact that the nonequilibrium state of material system turns out to be selfvarying. (It is evident that this selfvariation proceeds under the action of internal force whose quantity is described by the commutator of the unclosed evolutionary form ω^p.) State of material system changes but remains nonequilibrium during this process.

Since the evolutionary form is unclosed, the evolutionary relation cannot be identical. This means that the nonequilibrium state of material system holds. But in this case it is possible a transition of material system to a locally equilibrium state.

This follows from one more property of nonidentical evolutionary relation. Under selfvariation of the evolutionary relation it can be realized the conditions of degenerate transformation. And under degenerate transformation from the nonidentical relation it is obtained the identical relation.

From identical relation one can define the state differential pointing to the equilibrium state of the system. However, such system state is realized only locally due to the fact that the state differential obtained is an interior one defined only on pseudostructure, that is specified by the conditions of degenerate transformation. And yet the total state of material system remains to be nonequilibrium because the evolutionary relation, which describes the material system state, remains nonidentical one.

The conditions of degenerate transformation are connected with symmetries caused by degrees of freedom of material system. These are symmetries of the metric forms commutators of the manifold. {To the degenerate transformation it must correspond a vanishing of some functional expressions, such as Jacobians, determinants, the Poisson brackets, residues and others. Vanishing of these functional expressions is the closure condition for dual form. And it should be emphasize once more that the degenerate transformation is realized as a transition from the accompanying noninertial frame of reference to the locally inertial system. The evolutionary form and nonidentical evolutionary relation are defined in the noninertial frame of reference (deforming manifold). But the closed exterior form obtained and the identical relation are obtained with respect to the locally-inertial frame of reference (pseudostructure)}.

Realization of the conditions of degenerate transformation is a vanishing of the commutator of manifold metric form, that is, a vanishing of the dual form commutator. And this leads to realization of pseudostructure and formatting the closed inexact form, whose closure conditions have the form

$$d_\pi \omega^p = 0, d_\pi{}^* \omega^p = 0 \tag{7}$$

On the pseudostructure π from evolutionary relation (5) it is obtained the relation

$$d_\pi \psi = \omega_\pi^p \tag{8}$$

which proves to be an identical relation since the closed inexact form is a differential (interior on pseudostructure).

The realization of the conditions of degenerate transformation and obtaining identical relation from nonidentical one has both mathematical and physical

meaning. Firstly, this points to the fact that the solution of equations of balance conservation laws proves to be a generalized one. And secondly, from this relation one obtains the differential $d_\pi\psi$ and this points to the availability of the state-function (potential) and that the state of material system is in local equilibrium.

Relation (8) holds the duality. The left-hand side of relation (8) includes the differential, which specifies material system and whose availability points to the locally-equilibrium state of material system. And the right-hand side includes a closed inexact form, which is a characteristics of physical fields. The closure conditions (7) for exterior inexact form correspond to the conservation law, i.e. to a conservative on pseudostructure quantity, and describe a differential-geometrical structure. These are such structures (pseudostructures with conservative quantities) that are physical structures formatting physical fields[6,7].

The transition from nonidentical relation (5) obtained from equations of the balance conservation laws to identical relation (8) means the following. Firstly, an emergency of the closed (on pseudostructure) inexact exterior form (right-hand side of relation (8)) points to an origination of the physical structure. And, secondly, an existence of the state differential (left-hand side of relation (8)) points to a transition of the material system from nonequilibrium state to the locally-equilibrium state.

Thus one can see that the transition of material system from nonequilibrium state to locally-equilibrium state is accompanied by originating differential-geometrical structures, which are physical structures. Massless particles, charges, structures made up by eikonal surfaces and wave fronts, and so on are examples of physical structures.

The duality of identical relation also explains the duality of nonidentical evolutionary relation. On the one hand, evoltionary relation describes the evolutionary process in material systems, and on the other describes the process of generating physical fields.

The emergency of physical structures in the evolutionary process reveals in material system as an emergency of certain observable formations, which develop spontaneously. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, and others. It appears that structures of physical fields and the formations of material systems observed are a manifestation of the same phenomena. The light is an example of such a duality. The light manifests itself in the form of a massless particle (photon) and of a wave.

This duality also explains a distinction in studying the same phenomena in material systems and physical fields. In the physics of continuous media (material systems) the interest is expressed in generalized solutions of equations of the balance conservation laws. These are solutions that describe the formations in material media observed. The investigation of relevant physical structures is carried out using the field-theory equations.

The unique properties of nonidentical evolutionary relation, which describes

the connection between physical fields and material systems, discloses the connection of evolutionary relation with the field-theory equations. In fact, all equations of existing field theories are the analog to such relation or its differential or tensor representation.

2. Specific features of field-theory equations

The field-theory equations are equations that describe physical fields. Since physical fields are formatted by physical structures, which are described by closed exterior *inexact* forms and by closed dual forms (metric forms of manifold), is obvious that the field-theory equations or solutions to these equations have to be connected with closed exterior forms. Nonidentical relations for functionals like wave-function, action functional, entropy, and others, which are obtained from the equations for material media (and from which identical relations with closed forms describing physical fields are obtained), just disclose the specific features of the field-theory equations.

The equations of mechanics, as well as the equations of continuous media physics, are partial differential equations for desired functions like a velocity of particles (elements), temperature, pressure and density, which correspond to physical quantities of material systems (continuous media). Such functions describe the character of varying physical quantities of material system. The functionals (and state-functions) like wave-function, action functional, entropy and others, which specify the state of material systems, and corresponding relations are used in mechanics and continuous media physics only for analysis of integrability of these equations. And in field theories such relations play a role of equations. Here it reveals the duality of these relations. In mechanics and continuous media physics these equations describe the state of material systems, whereas in field-theory they describe physical structures from which physical fields are formatted.

It can be shown that all equations of existing field theories are in essence relations that connect skew-symmetric forms or their analogs (differential or tensor ones). And yet the nonidentical relations are treated as equations from which it can be found identical relation with include closed forms describing physical structures desired.

Field equations (the equations of the Hamilton formalism) reduce to identical relation with exterior form of first degree, namely, to the Poincare invariant

$$ds \,=\, -\,H\,dt \,+\, p_j\,dq_j \tag{9}$$

{As is known, the field equation has the form

$$\frac{\partial s}{\partial t} + H\left(t,\, q_j,\, \frac{\partial s}{\partial q_j}\right) \,=\, 0, \quad \frac{\partial s}{\partial q_j} \,=\, p_j \tag{10}$$

here s is a field function for the action functional $S \,=\, \int L\,dt$. Here L is the Lagrangian function, $H(t,\, q_j,\, p_j) \,=\, p_j\,\dot{q}_j - L$ is the Hamilton function $p_j \,=\, \partial L/\partial \dot{q}_j$. These functions satisfy the relations:

$$\frac{dg_j}{dt} \,=\, \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} \,=\, -\frac{\partial H}{\partial g_j} \tag{11}$$

Relations (11), which present a set of the Hamilton equations, are the closure conditions for exterior and dual forms [6].}

The Schrődinger equation in quantum mechanics is an analog to field equation, where the conjugated coordinates are replaced by operators. The Heisenberg equation corresponds to the closure condition of dual form of zero degree. Dirac's *bra-* and *cket-* vectors made up a closed exterior form of zero degree. It is evident that the relations with skew-symmetric differential forms of zero degree correspond to quantum mechanics. The properties of skew-symmetric differential forms of the second degree lie at the basis of the electromagnetic field equations. The Maxwell equations may be written as $d\theta^2 = 0$, $d^*\theta^2 = 0$, where $\theta^2 = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ (here $F_{\mu\nu}$ is the strength tensor). The Einstein equation is a relation in differential forms. This equation relates the differential of dual form of first degree (Einstein's tensor) and a closed form of second degree –the energy-momentum tensor. (It can be noted that, Einstein's equation is obtained from differential forms of third degree).

The connection the field theory equations with skew-symmetric forms of appropriate degrees shows that there exists a commonness between field theories describing physical fields of different types. This can serve as an approach to constructing the unified field theory. This connection shows that it is possible to introduce a classification of physical fields according to the degree of skew-symmetric differential forms. From relations (5) and (8) one can see that relevant degree of skew-symmetric differential forms, which can serve as a parameter of unified field theory, is connected with the degree p of evolutionary form in relation (5). It should be noted that the degree p is connected with the number of interacting balance conservation laws. {The degree of closed forms also reflects a type of interaction [7]. Zero degree is assigned to a strong interaction, the first one does to a weak interaction, the second one does to electromagnetic interactions, and the third degree is assigned to gravitational field.}

The connection of field-theory equations, which describe physical fields, with the equations for material media discloses the foundations of the general field theory. As an equation of general field theory it can serve the evolutionary relation (5), which is obtained from equations the balance conservation laws for material system and has a double meaning. On the one hand, that, being a relation, specifies the type of solutions to equations of balance conservation laws and describes the state of material system (since it includes the state differential), and, from other hand, that can play a role of equations for description of physical fields (for finding the closed inexact forms, which describe the physical structures from which physical fields are made up). It is just a double meaning that discloses the connection of physical fields with material media (which is based on the conservation laws) and allows to understand on what the general field theory has to be based.

In conclusion it should be emphasized that the study of equations of mathematical physics appears to be possible due to unique properties of skew-symmetric differential forms. In this case, beside the exterior skew-symmetric differential forms, which are defined on differentiable manifolds, the skew-symmetric dif-

ferential forms, which, unlike to the exterior forms, are defined on deforming (nondifferentiable) manifolds [6], were used.

References

1. Tolman R. C., Relativity, Thermodynamics, and Cosmology. Clarendon Press, Oxford, UK, 1969.

2. Fock V. A., Theory of space, time, and gravitation. -Moscow, Tech. Theor. Lit., 1955 (in Russian).

3. Clark J. F., Machesney M., The Dynamics of Real Gases. Butterworths, London, 1964.

4. Cartan E., Les Systemes Differentials Exterieus ef Leurs Application Geometriques. -Paris, Hermann, 1945.

5. Schutz B. F., Geometrical Methods of Mathematical Physics. Cambrige University Press, Cambrige, 1982.

6. Petrova L. I. Skew-symmetric differential forms: Conservation laws. Foundations of field theories. -Moscow, URSS, 2006, 158 p. (in Russian).

7. Petrova L. I. The quantum character of physical fields. Foundations of field theories, Electronic Journal of Theoretical Physics, v.3, 10 (2006), 89-107p.